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NUMERICAL ANALYSIS OF SOME EXTERIOR PROBLEMS, MIXED METHODS AND A POSTERIORI ERROR ANALYSIS IN FLUID MECHANICS AND ELASTICITY

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Abstract

This paper contains a brief description of various problems in the field of numerical analysis of partial differential equations. First, some results concerning the numerical analysis of boundary value problems in exterior domains of the plane are reviewed. Then, the derivation of dual-dual mixed variational formulations in fluid mechanics is explored. Finally, some results related to a posteriori error analysis in linear elasticity and a new augmented formulation in elasticity are discussed.

Key words: *Finite element method, boundary element method, symmetric coupling, mixed finite elements, twofold saddle point formulation, augmented formulation, a posteriori error analysis, parabolic-elliptic problem, quasi-linear problem, quasi-Newtonian flow, Stokes equation, linear elasticity, hyperelasticity.*

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1 Introduction

The aim of our research is to design and analyze efficient numerical methods that could be used to solve boundary value problems for partial differential equations in practice. We have dealt with problems in exterior domains, including linear and nonlinear elliptic and parabolic-elliptic equations and some problems in elasticity. We also coped with some models from fluid mechanics and elasticity in bounded domains.

The numerical solution of boundary value problems in exterior domains combining boundary elements and finite elements present several difficulties

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in practice. Indeed, these methods lead to ill-conditioned and badly structured systems of equations, and the computation of the boundary terms is cumbersome when the coupling boundary is a polygonal curve. Moreover, in this case we do not know how to control the effect of numerical quadratures on convergence. These facts motivated the work described in section 2. We present there a parametrized version of the standard symmetric method of coupling boundary elements and finite elements. This technique offers some advantages to solve a class of problems posed in exterior domains of the plane and has been applied to several nonlinear problems and to the elasticity system (cf. [43, 58, 57, 44]).

Later, we became interested in the derivation of mixed methods in fluids mechanics when the constitutive law cannot be inverted explicitly. Dual-dual methods are of special interest in this situation since no inversion process is required in their derivation. They had been applied to some linear and nonlinear problems in potential theory and elasticity (cf. [32, 35, 3, 40]). In [29] we derived a low-order mixed finite element method based on a dual-dual formulation for a class of quasi-Newtonian Stokes flows. In particular, we obtained, as a by-product, a new mixed finite element method for the usual Stokes problem. We carried out an a posteriori error analysis, based on local problems, and obtained fully explicit and reliable a posteriori error estimates of the accuracy of the computed numerical solution (see [30]). Then, we applied the approach from [29] to derive a low-order mixed finite element method for the generalized Stokes problem and developed the corresponding a posteriori error analysis (see [11]). In section 3, we review the derivation of dual-dual mixed formulations for quasi-Newtonian Stokes flows and for the generalized Stokes problem.

More recently, we turned our attention to an augmented mixed finite element method proposed in [27] for the linear elasticity system in the plane. We developed a residual-based a posteriori error analysis (see [4, 5]), combining a technique used in mixed finite element schemes with the usual procedure applied to primal finite element methods, and obtained a posteriori error estimators of residual type that are both efficient and reliable. In the last section, we outline the proof in the case of pure homogeneous Dirichlet boundary conditions.

2 Symmetric coupling of boundary elements and finite elements for solving exterior problems in 2D

Many physical and engineering problems are naturally posed in the exterior of a bounded domain; typical applications arise in electromagnetism and acoustics. The finite element method (FEM) can be used to solve nonlinear and nonhomogeneous boundary value problems; however, it can only be applied in bounded regions. On the other hand, the boundary element method (BEM) is well suited to solve problems in unbounded domains since it is based on the idea of reducing the partial differential equation to an integral equation on the boundary, yet it has the drawback that the equation must be linear, homogeneous and with constant coefficients. BEM-FEM methods were

conceived with the aim of making the most of both techniques to solve boundary value problems in exterior domains (cf. for instance [74, 75]).

When a boundary value problem in an exterior domain is solved using a BEM-FEM method, an artificial boundary –called *the coupling boundary*– is introduced in order to divide the domain of the original problem in two regions: a bounded interior region and an unbounded exterior one, so that the equation is linear, homogeneous and with constant coefficients in the latter. Then, the problem can be written equivalently as a *transmission problem*, demanding that the solution satisfy appropriate conditions on the coupling boundary. Next, the BEM is applied in the unbounded region and the problem there is reduced to an integral equation on the coupling boundary. Then, the original problem reduces to a problem in the bounded region with *non-local* boundary conditions on the coupling boundary, and can be solved using the FEM. Finally, the solution in the exterior region is recovered through the integral representation formula.

To fix ideas, let us consider the Poisson problem in an exterior domain of $\mathbb{R}^2$. Let $\Omega_0 \subset \mathbb{R}^2$ be a bounded domain. We assume, for simplicity, that $\Gamma_0 := \partial\Omega_0$ is a polygonal curve, and denote by $\Omega_0' := \mathbb{R}^2 \setminus \overline{\Omega}_0$. Given $f \in L^2(\Omega_0')$, of compact support, we look for a function $u\colon \Omega_0' \to \mathbb{R}$ such that

$$\left\{ \begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega_0', \\ u & = & 0 & \text{on } \Gamma_0, \\ u & = & \mathcal{O}(1) & \text{as } |\mathbf{x}| \to +\infty. \end{array} \right. \tag{1}$$

Let Γ be a simple closed curve such that the support of f and the domain $\overline{\Omega}_0$ are contained in the region bounded by Γ. The coupling boundary Γ divides the domain of the original problem, Ω_0', in two regions: a bounded interior region, that we denote Ω^-, and the unbounded region exterior to Γ, that we denote Ω^+. The limit or trace over Γ of a function v defined in Ω^+ (resp., Ω^-) is denoted

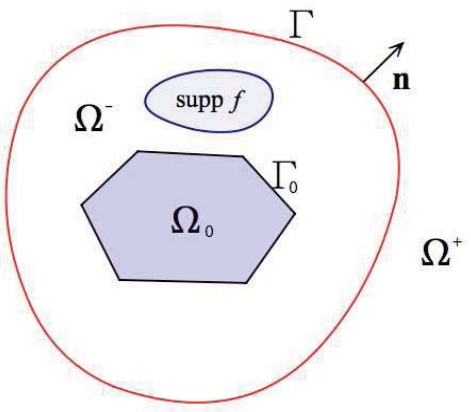

Figure 1: Domain of the transmission problem

by v^+ (resp., v^-). Finally, $\mathbf{n}$ denotes the unit normal vector to Γ, pointing from

Ω^- to Ω^+, and $\mathbf{t}$ denotes the tangent vector.

Then, problem (1) is equivalent to a transmission problem, which consists of a problem posed in Ω^-:

$$\left\{ \begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega^-, \\ u & = & 0 & \text{on } \Gamma_0, \end{array} \right. \tag{2}$$

and an *homogeneous* problem in Ω^+:

$$\left\{ \begin{array}{rcll} -\Delta u & = & 0 & \text{in } \Omega^+, \\ u & = & \mathcal{O}(1) & \text{as } |\mathbf{x}| \to +\infty, \end{array} \right. \tag{3}$$

coupled by means of *transmission conditions* on the coupling boundary Γ:

$$u^- = u^+, \qquad \frac{\partial u}{\partial \mathbf{n}}^- = \frac{\partial u}{\partial \mathbf{n}}^+ . \tag{4}$$

The variational formulation of problem (2) reads: find $u \in V$ such that

$$a(u,v) - \int_\Gamma \frac{\partial u}{\partial \mathbf{n}}^- v^- = \int_{\Omega^-} fv \qquad \forall\, v \in V, \tag{5}$$

where $V := \{v \in H^1(\Omega^-) : v|_{\Gamma_0} = 0\}$ and $a(u,v) := \displaystyle\int_{\Omega^-} \nabla u \cdot \nabla v\,.$

On the other hand, Green's formula applied in Ω^+ to the solution to problem (3) and the fundamental solution of the bidimensional Laplacian, $G(\mathbf{x},\mathbf{y}) := -\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}|$, yields the following integral representation formula for u in Ω^+ (cf. for instance [19]):

$$u(\mathbf{x}) = \int_\Gamma u^+(\mathbf{y}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})}\, d\sigma_{\mathbf{y}} - \int_\Gamma G(\mathbf{x},\mathbf{y})\, \frac{\partial u}{\partial \mathbf{n}}^+ (\mathbf{y})\, d\sigma_{\mathbf{y}} + u_\infty \quad \forall\, \mathbf{x} \in \Omega^+. \tag{6}$$

The first integral in (6) stands for the double layer potential with density u^+ whereas the second integral represents the single layer potential with density $\frac{\partial u}{\partial \mathbf{n}}^+$. The constant u_∞ accounts for the asymptotic behavior of u at infinity and can be computed from the trace of u and its normal derivative on Γ (see [64]).

We remark that, once we know u^+ and $\frac{\partial u}{\partial \mathbf{n}}^+$, the solution to problem (3) can be computed in any point of Ω^+ using (6). Therefore, taking into account the transmission conditions (4), to solve problem (1) it is enough to determine the solution in the bounded domain Ω^- and its normal derivative $\frac{\partial u}{\partial \mathbf{n}}^-$. We need an additional condition relating these two unknowns. A usual choice is to consider the integral equation derived from (6) as $\mathbf{x}$ tends to Γ (cf. [19]):

$$\frac{1}{2}\, u = \widehat{\mathcal{K}} u - \widehat{\mathcal{V}} \frac{\partial u}{\partial \mathbf{n}} + u_\infty \qquad \text{on } \Gamma, \tag{7}$$

where $\widehat{\mathcal{K}}$ and $\widehat{\mathcal{V}}$ denote, respectively, the double layer and the single layer operators, formally defined as the corresponding potentials. On the other hand,

the asymptotic behavior of u at infinity and (6) imply that $\int_\Gamma \frac{\partial u}{\partial \mathbf{n}}\, d\sigma = 0$. Then, the integral equation (7) can be tested with functions of zero mean on Γ, so that the constant u_∞ disappears (it can be recovered testing (7) with a constant function).

Equations (5) and (7) are the basis of the BEM-FEM method analyzed in [48] by C. Johnson and J.-C. Nédélec. Until very recently (see [65]), it was thought that this method only worked well if the double layer operator $\widehat{\mathcal{K}}$ was compact. As a result, the choice of a smooth coupling boundary Γ was mandatory for the Laplace equation. In other applications, such as linear elasticity (where the double layer operator is never compact), this formulation is not used. In addition, this procedure leads to nonsymmetric systems of linear equations. For these reasons, M. Costabel [21] and H.D. Han [46] introduced the *symmetric method* of coupling boundary elements and finite elements. This method is based on adding a second integral equation on the coupling boundary:

$$\frac{1}{2}\frac{\partial u}{\partial \mathbf{n}} = -\widehat{\mathcal{W}} u - \widehat{\mathcal{K}}^* \frac{\partial u}{\partial \mathbf{n}} \qquad \text{on } \Gamma,$$

where $\widehat{\mathcal{W}}$ is the *hypersingular* operator and $\widehat{\mathcal{K}}^*$ is the adjoint of the double layer operator. We recall from [49] that, for the Laplacian, the hypersingular operator can be expressed in terms of the single layer operator as follows:

$$\widehat{\mathcal{W}} = -\frac{\partial}{\partial \mathbf{t}} \widehat{\mathcal{V}} \frac{\partial}{\partial \mathbf{t}}\,.$$

In the symmetric method, the compactness of the double layer operator does not play any role. Then, it is possible to choose a polygonal curve as coupling boundary and this is, in fact, what all the authors do. However, this choice leads to additional difficulties in the approximation of the boundary terms, that include integrals with singular kernels. Moreover, in this case we do not know how to analyze the effect of numerical quadrature on convergence.

In [43], we followed [56] and choose a smooth coupling boundary Γ. Using a parametrization of Γ, we obtain a new version of the symmetric method for exterior boundary value problems in the plane. The new formulation is equivalent to the standard symmetric method introduced in [21, 46], but allows to approximate the singular integrals from the boundary element method using only low order quadrature formulas. In addition, with this approach it is possible to analyze the effect of numerical quadrature on convergence, which is in fact the main contribution in [43].

In what follows, we assume that Γ is of class $\mathcal{C}^\infty$ and let $\boldsymbol{x}\colon \mathbb{R} \to \mathbb{R}^2$ be a 1-periodic parametrization of Γ. We consider the 1-periodic Sobolev space of index 1/2, defined by

$$H^{1/2} := \Big\{ \phi \in L^2[0,1] \,:\, \sum_{m=-\infty}^{\infty} (1+m^2)^{1/2}\, |\hat{\phi}(m)|^2 < \infty \Big\},$$

where $\hat{\phi}(m) := \int_0^1 \phi(s)\, e^{-2\pi i m s}\, ds$, for $m \in \mathbb{Z}$, are the Fourier coefficients of ϕ. We denote by $H^{-1/2}$ the dual space of $H^{1/2}$, and by $(\cdot,\cdot)$ the duality product

between $H^{-1/2}$ and $H^{1/2}$. We consider parametrized versions, $\mathcal{V}$ and $\mathcal{K}$, of the single and double layer operators (cf. [43, Section 1.2]), that inherit the properties of the standard ones. Indeed, $\mathcal{V}\colon H^{-1/2} \to H^{1/2}$ is continuous and elliptic on $H_0^{-1/2} := \{\eta \in H^{-1/2} : (\eta, 1) = 0\}$, and $\mathcal{K}\colon H^{1/2} \to H^{1/2}$ is compact.

Then, introducing the parametrization $\boldsymbol{x}$ in the integrals over Γ that appear in the symmetric method, we derive a new continuous BEM-FEM formulation, equivalent to the one introduced in [21, 46]:

$$\left\{\begin{array}{l} \text{find } u \in V \text{ and } \xi \in H_0^{-1/2} \text{ such that} \\ \quad a(u,v) + d(u,v) - c(v,\xi) \;=\; (f,v)_{L^2(\Omega^-)} \qquad \forall\, v \in V \\ \qquad\qquad c(u,\eta) + b(\xi,\eta) \;=\; 0 \qquad\qquad \forall\, \eta \in H_0^{-1/2} \end{array}\right. \tag{8}$$

where, for simplicity, we substitute the unknown $\frac{\partial u}{\partial \mathbf{n}}^+$ by $\xi := |\boldsymbol{x}'| \left(\frac{\partial u}{\partial \mathbf{n}}^+ \circ \boldsymbol{x}\right)$, and for any $\xi, \eta \in H^{-1/2}$ and $u, v \in H^1(\Omega^-)$, define $b(\xi,\eta) := (\eta, \mathcal{V}\xi)$, $d(u,v) := b(\gamma(u)', \gamma(v)')$ and

$$c(v,\eta) := \left(\eta, \left(\frac{1}{2}\,\mathcal{I} - \mathcal{K}\right)\gamma(v)\right),$$

where $\gamma : H^1(\Omega^-) \to H^{1/2}$ is the parametrized trace, that extends the map $u \mapsto \gamma(u) := u|_\Gamma \circ \boldsymbol{x}$. Existence and uniqueness of a solution to (8) are a consequence of Lax–Milgram's Lemma and the properties of the single and double layer operators.

In [43] the discrete problem is defined using a regular family of exact triangulations $\{\mathcal{T}_h^-\}_h$ of the bounded domain $\overline{\Omega^-}$ (cf. [73]), that contains straight triangles and triangles with exactly one curved side (the one that fits the coupling boundary). The corresponding finite element subspaces, $V_h \subset V$, are defined using Lagrange curved finite elements of order one on the curved triangles combined with Lagrange finite elements of the same order over the straight triangles, so that global finite element functions are continuous in $\overline{\Omega^-}$. To approximate the unknown ξ, a family of subspaces $H_h \subset H_0^{-1/2}$ consisting of 1-periodic splines of order one over a uniform partition of the real line, $s_i := ih$, $i \in \mathbb{Z}$, are used. Using interpolation error bounds on curved triangles and an approximation result from [68] we derive optimal error estimates. We remark that this method can be generalized without any difficulty to higher order approximations. It is also possible to consider different approximating functions on an independent mesh of the boundary; for instance, we could use trigonometric polynomials (see [59, 60]). On the other hand, the use of ideal triangles could be avoided (see [64]).

In practice, it is not possible to compute exactly some coefficients of the linear system obtained from the discretization process and it is necessary the use of quadrature formulas. We describe next the quadrature formulas used to approximate the integrals of the discrete problem:

- *Integrals over a triangle $T \in \mathcal{T}_h^-$.* We consider a quadrature formula of order zero over a reference (straight) triangle and define the corresponding formula over a triangle $T \in \mathcal{T}_h^-$ through a change of variables. In this way, we define approximations $a_h(\cdot,\cdot)$ and $l_h(\cdot)$ of the bilinear form $a(\cdot,\cdot)$ and the linear form $l(\cdot) := (f,\cdot)_{L^2(\Omega^-)}$, respectively.

- *Approximation of the bilinear form $b(\cdot,\cdot)$.* The single layer operator, $\mathcal{V}$, shows a singularity of logarithmic type. Then, to approximate the associated bilinear form, $b(\cdot,\cdot)$, we decompose the integrand as in G.C. Hsiao et al. [47], so that it can be written as a sum of a $\mathcal{C}^\infty$ function,

$$F(s,t) := \begin{cases} \log \dfrac{|\boldsymbol{x}(s)-\boldsymbol{x}(t)|^2}{(s-t)^2} & \text{if } s \neq t, \\ \log |\boldsymbol{x}'(s)|^2 & \text{if } s = t, \end{cases}$$

and a term that can be computed exactly. Let $\hat{\ell}_2$ be a quadrature formula of order one on the unit square. Then, the bilinear form $b_h\colon H_h \times H_h \to \mathbb{R}$ is defined by

$$b_h(\xi_h,\eta_h) := \sum_{i,j=1}^{N} \xi_h|_{(s_{i-1},s_i)}\eta_h|_{(s_{i-1},s_i)}\tilde{b}_{i,j} \quad \forall\, \xi_h,\eta_h \in H_h,$$

where, for $i,j = 1,\ldots,N$,

$$\tilde{b}_{i,j} := -\frac{1}{4\pi}\, h^2\, (\hat{\ell}_2(F(s_{\underline{i}-1} + h\,\cdot, s_{\underline{j}-1} + h\,\cdot)) + \log h^2 + B_{\underline{i}-\underline{j}}),$$

with

$$(\underline{i},\underline{j}) := \begin{cases} (i,j) & \text{if } |i-j| \leq N/2, \\ (i,j+N) & \text{if } i-j > N/2, \\ (i,j-N) & \text{if } j-i > N/2, \end{cases}$$

and $B_0 = -3$, $B_1 = 4\log(2) - 3$ and for $k \geq 2$,

$$B_k = 2\log(k) - \sum_{n=1}^{\infty} \frac{1}{n(n+1)(2n+1)}\,\frac{1}{k^{2n}}\,.$$

- *Approximation of the bilinear form $d(\cdot,\cdot)$.* We define the bilinear form $d_h\colon V_h \times V_h \to \mathbb{R}$ by

$$d_h(u_h,v_h) := \sum_{i,j=1}^{N} \gamma u(s_i)\,\gamma v(s_j)\,\tilde{d}_{i,j} \quad \forall\, u_h,v_h \in V_h,$$

where, for $i,j = 1,\ldots,N$, $\tilde{d}_{i,j} := \big(\tilde{b}_{i,j} - \tilde{b}_{i,j+1} - \tilde{b}_{i+1,j} + \tilde{b}_{i+1,j+1}\big)/h^2$.

- *Approximation of the bilinear form $c(\cdot,\cdot)$.* Let $v_h \in V_h$ and $\eta_h \in H_h$. We remark that $\gamma(v_h) \in T_h$, where

$$T_h := \{\eta_h \in \mathcal{C}(\mathbb{R})\ ;\ \eta_h \text{ 1-periodic and } \quad \eta_h|_{(s_{i-1},s_i)} \in \mathcal{P}_1 \quad \forall\, i \in \mathbb{Z}\}.$$

Let $\{l_i\}_{i=1}^N$ be the nodal basis of T_h. We compute $(\eta_h, \gamma v_h)$ exactly, so that it only remains to approximate the coefficients

$$c_{i,j} := \int_{s_{j-1}}^{s_j} \left(\int_{s_{i-1}}^{s_{i+1}} K(s,t) l_i(t)\, dt \right) ds \qquad i,j = 1,\ldots,N,$$

where $K(\cdot,\cdot)$ is the kernel of the double layer operator $\mathcal{K}$. Since $K(\cdot,\cdot)$ is a function of class $\mathcal{C}^\infty$, we can use $\hat{\ell}_2$ to define the approximations

$$\tilde{c}_{i,j} := h^2 \hat{\ell}_2\big(K(s_{j-1}+h\cdot, s_{i-1}+h\cdot) l_i(s_{i-1}+h\cdot) + K(s_{j-1}+h\cdot, s_i+h\cdot) l_i(s_i+h\cdot)\big),$$

Then, we define the bilinear form $c_h \colon V_h \times H_h \to \mathbb{R}$ by

$$c_h(v_h,\eta_h) := \frac{h}{4} \sum_{j=1}^{N} \eta_j \big(\gamma v(s_{j-1}) + \gamma v(s_j)\big) - \sum_{i,j=1}^{N} \eta_j \gamma v(s_i) \tilde{c}_{i,j}\,.$$

We proved in [43] that the fully discrete scheme based on these approximations:

$$\left\{ \begin{array}{l} \text{find } u_h^* \in V_h \text{ and } \xi_h^* \in H_h \text{ such that} \\ \begin{array}{rcll} a_h(u_h^*, v_h) + d_h(u_h^*, v_h) - c_h(v_h, \xi_h^*) & = & l_h(v_h) & \forall\, v_h \in V_h \\ c_h(u_h^*, \eta_h) + b_h(\xi_h^*, \eta_h) & = & 0 & \forall\, \eta_h \in H_h \end{array} \end{array} \right. \tag{9}$$

is well posed for h sufficiently small and derived optimal error estimates. More precisely, if $f \in W^{1,\infty}(\Omega^-)$ and $u \in H^2(\Omega^-)$, then there exists a constant $C > 0$, independent of h, such that

$$\|u - u_h^*\|_{H^1(\Omega^-)} + \|\xi - \xi_h^*\|_{H^{-1/2}} \le Ch \left(\|u\|_{H^2(\Omega^-)} + \|f\|_{W^{1,\infty}(\Omega^-)} \right).$$

The fully discrete scheme (9) can be implemented in the computer directly. However, this method leads to a system of equations of the form:

$$\begin{pmatrix} A + D & C^t \\ C & -B \end{pmatrix} \begin{pmatrix} \mathbf{u}_h^* \\ \boldsymbol{\xi}_h^* \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}, \tag{10}$$

where A and B are symmetric and positive definite matrices, and D is symmetric and semidefinite; moreover, A is a sparse matrix whereas B, C and D are dense matrices. Therefore, one has to solve a symmetric indefinite linear system of equations which, in addition, is ill-conditioned and badly structured. The use of an efficient iterative solver is then necessary.

We proposed an algorithm based on a preconditioning technique due to J.H. Bramble and J.E. Pasciak [7]. The idea is to transform the original system (10) into an equivalent one, with a symmetric and positive definite matrix in a certain inner product. Indeed, let R be a preconditioner of matrix A. Then, the system of linear equations

$$\begin{pmatrix} R^{-1}(A+D) & R^{-1}C^t \\ CR^{-1}(A+D)-C & B+CR^{-1}C^t \end{pmatrix} \begin{pmatrix} \mathbf{u}_h^* \\ \boldsymbol{\xi}_h^* \end{pmatrix} = \begin{pmatrix} R^{-1}\mathbf{f} \\ CR^{-1}\mathbf{f} \end{pmatrix}, \tag{11}$$

is equivalent to system (10). We proved that the matrix of the linear system (11) is symmetric and positive definite in the inner product

$$\left[\begin{pmatrix} u_h \\ \xi_h \end{pmatrix}, \begin{pmatrix} v_h \\ \eta_h \end{pmatrix} \right] := ((A+D-R)u_h, v_h)_{L^2(\Omega^-)} + (\xi_h, \eta_h).$$

Therefore, (11) can be solved by a preconditioned conjugate gradient method in the inner product $[\cdot,\cdot]$. Moreover, using Theorem 1 in [7], we showed that (11) can be preconditioned easily using only a preconditioner P of B (cf. [43] for more details).

This technique allows to *uncouple* the problem, since in each iteration we solve independently a problem using BEM and another problem using FEM. The method requires two preconditioners, one for the FEM stiffness matrix A and another one for the matrix associated with the single layer operator, B. Numerical experiments confirm the theoretical results and show that the algorithm is optimal in the sense that the number of iterations is independent of the discretization parameter.

Extension to nonlinear elliptic problems. The standard symmetric method was applied successfully to nonlinear boundary value problems that become homogeneous and linear with constant coefficients outside a bounded region (cf. [23, 37]). In these extensions, the error analysis is done assuming that the nonlinear operator is strongly monotone and Lipschitz-continuous, since in this case a Céa-type estimate is available. The parametrized version of the symmetric method can be extended fairly straightforward to this case (cf. [43]). The analysis of the continuous problem follows [37] and is based on the theory of monotone operators and Banach Fixed Point Theorem. The analysis of the discrete problem, based on a Céa type estimate, required to prove a technical result on the approximation in Sobolev spaces of non-integer index (which is a generalization of a result given in [70]). In addition, we analyzed a fully discrete scheme defined using only low order quadrature formulas. The analysis relies on a Strang-type inequality and on the analysis of the effect of numerical quadratures in the FEM for a nonlinear equation (cf. [25]).

On the other hand, J. Xu [71] introduced a technique to carry out the numerical analysis of nonlinear problems in bounded domains without using a Céa estimate. The idea is to linearize the nonlinear partial differential equation around an isolated solution and consider the finite element discretization of the linearized problem. In [58] we extended this technique to analyze exterior

nonlinear problems without using discrete Green functions, at the expense of certain restrictions in the type of nonlinearity. We cannot deal with the general case because we do not know bounds for discrete Green functions associated with BEM–FEM formulations. Existence of a solution to the discrete problem and error estimates are derived applying Brouwer's Fixed Point Theorem; local uniqueness is also proved. The main contribution is the analysis of a fully discrete nonlinear BEM–FEM formulation without using Strang's lemma. We proved that the method described in [71] can be completed in this case to study the effect of numerical quadratures on convergence. This question remained open even in the bounded case and still is for a general nonlinear equation.

Nonlinear parabolic-elliptic problems. In [22], M. Costabel et al. applied the symmetric method to an exterior linear parabolic-elliptic problem. They used the Crank–Nicolson method for the time discretization and proved convergence of the solutions to the discrete schemes and theoretical error estimates. In [44], we applied the parametrized version of the symmetric method to a nonlinear parabolic-elliptic problem in the plane. This kind of problems appears in the modeling of quasi-stationary electromagnetic fields. The discrete problem is defined using the backward Euler method for the time discretization and an exact triangulation of the bounded domain. The analysis follows essentially [72]: existence, uniqueness, convergence of the discrete solutions and optimal error estimates are derived assuming that the nonlinear operator is strongly monotone and Lipschitz-continuous. In addition, we proposed a fully discrete scheme using only quadrature formulas of low order and, under some additional conditions on the nonlinear operator, proved that the order of convergence is optimal.

Extensions to elasticity. In [23, 36], the symmetric method was applied to a problem of three-dimensional elasticity theory, where an elastoplastic material is embedded into a linear elastic material. In two dimensions, this problem was analyzed in [16]. In [57] we generalized the parametrized version of the symmetric method to study an homogeneous isotropic linear elastic material in an exterior domain of the plane. In this case, the solution in the exterior region is given by Betti-Somigliana's formula (cf. [19]), and can be computed once we know the values of the solution and its traction on the coupling boundary. We solved the difficulties that result from the singularities of the integral operators and proposed a fully discrete scheme based on the use of quadrature formulas of low order. Optimal error estimates are derived and a preconditioning technique based on that of [7] is suggested to solve the corresponding linear systems. In [43] we showed that this technique can also be applied to the problem considered in [16] and proposed a fully discrete scheme that entails a great computational saving.

3 Dual-dual mixed methods in fluid mechanics

In this section we recall a dual-dual mixed finite element method to solve a class of quasi-Newtonian Stokes flows and discuss its application to the generalized Stokes problem. Mixed finite element methods are widely used to solve boundary value problems because they allow to approximate unknowns of physical interest directly. The standard mixed finite element method introduces the flux as an additional unknown; then, the gradient is expressed in terms of the flux and an integration by parts is done (cf. for instance [8, 42]).

When the constitutive law cannot be inverted explicitly, two basic strategies are available to obtain a mixed formulation. One possibility consists in inverting the constitutive law using the implicit function theorem (cf. [61, 62, 51]). The other strategy is based on introducing additional unknowns (preferably of physical interest) and rewriting the problem as a twofold saddle point operator equation, that is, the left-hand-side of the operator equation shows the following structure:

$$\begin{pmatrix} A & B^* \\ B & O \end{pmatrix} \qquad \text{with} \quad A = \begin{pmatrix} A_1 & B_1^* \\ B_1 & O \end{pmatrix}, \tag{12}$$

where B and B_1 are linear bounded operators and A_1 is a nonlinear operator. This kind of formulations are called *dual–dual* formulations. It is important to emphasize that no inversion process is required in their derivation, which constitutes one of their main advantages.

Although the structure of (12) is very similar to the standard one, results from [52, 53, 66] cannot be applied. Fortunately, the standard theory of Babuška–Brezzi was generalized in [26] to deal with this kind of problems (see also [35]). On the other hand, this type of formulations leads to the solution of linear systems with a twofold saddle point structure, which are symmetric, indefinite and ill-conditioned. Efficient iterative solvers are already available (see [34, 32, 33]).

Dual-dual formulations were introduced in the context of coupling mixed finite elements and boundary elements (cf. [31, 39, 33, 26]). The first dual-dual method for a finite element discretization was analyzed in [32], where a linear second-order elliptic equation in divergence form is considered and, besides the scalar unknown u and the flux, the gradient ∇u is introduced as a third explicit unknown[1]. Then, this technique was applied in [35] to obtain a fully discrete dual-dual formulation of a nonlinear elliptic problem in divergence form. Dual-dual formulations were also used in nonlinear elasticity, where the strain tensor is introduced as additional unknown (cf. [3] for a dual-dual formulation of a hyperelastic material and [40] for the incompressible case).

Concerning the derivation of dual-dual formulations in fluid mechanics, we considered in [29] a class of quasi-Newtonian Stokes flows, and extended the

[1]The idea of introducing the gradient as an additional unknown was suggested by G.N. Gatica and W.L. Wendland [41] in the context of coupling mixed finite elements and boundary elements. It was also used in T. Arbogast et al. [1] and in Z. Chen [17, 18], where it was called *expanded mixed finite element method*. Additional variables are also used in least-squares finite element methods (see, e.g. [13]).

procedure in [11] to the generalized Stokes problem. The mixed finite element methods proposed in [29, 11] simply rely on the introduction of the flux and the tensor gradient of the velocity as additional unknowns. Then, the variational formulation is written as a twofold saddle point operator equation, so that the abstract theory developed in [26] can be applied to prove that the continuous and discrete schemes are well posed. In particular, we showed that the stability of the Galerkin schemes can be ensured using only low-order finite element subspaces. We remark that the usual Stokes equations are included in the class of problems considered in [29], so that we obtained, as a by-product, a new mixed finite element method for the Stokes problem.

Next we describe the derivation of a low-order mixed FEM based on a dual-dual formulation for quasi-Newtonian Stokes flows. We let Ω be a bounded domain in $\mathbb{R}^2$ with a Lipschitz-continuous boundary Γ, and consider a nonlinear Stokes fluid occupying the region Ω under the action of an external force. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be the nonlinear kinematic viscosity function of the fluid. Given $\mathbf{f} \in [L^2(\Omega)]^2$ and $\mathbf{g} \in [H^{1/2}(\Gamma)]^2$, we look for the velocity $\mathbf{u} := (u_1, u_2)^{\mathtt{t}}$ and the pressure p such that:

$$\left\{ \begin{array}{rcll} -\,\mathbf{div}\,(\,\psi(|\nabla \mathbf{u}|)\,\nabla \mathbf{u} \,-\, p\,\mathbf{I}\,) & = & \mathbf{f} & \text{in } \Omega, \\ \mathrm{div}(\mathbf{u}) & = & 0 & \text{in } \Omega, \\ \mathbf{u} & = & \mathbf{g} & \text{on } \Gamma. \end{array} \right. \tag{13}$$

We recall that the Dirichlet datum $\mathbf{g}$ must satisfy the compatibility condition

$$\int_\Gamma \mathbf{g} \cdot \mathbf{n}\, ds = 0, \tag{14}$$

where $\mathbf{n}$ is the unit outward normal to Γ.

This kind of nonlinear Stokes problem arises in the modeling of a large class of non-Newtonian fluids (biological fluids, lubricants, paints and polymeric fluids among others). In particular, the Ladyzhenskaya law for fluids with large stresses (see [50]), the power law used to model many polimeric solutions and melts (see [45]), and the Carreau law, used to model viscoplastic flows and creeping flow of metals (see, e.g. [54, 67]), are included in this framework. For the nonlinear model satisfying the power law, a dual-mixed variational formulation based on inverting the relation $\tilde{\boldsymbol{\sigma}} = \psi(|\nabla \mathbf{u}|)\nabla \mathbf{u}$ to obtain $\nabla \mathbf{u}$ as an explicit function of $\tilde{\boldsymbol{\sigma}}$ was studied in [55]. However, this procedure cannot be applied to the Carreau law since in this case such explicit inversion formula is not available.

To derive a dual-dual mixed variational formulation for the boundary value problem (13), we introduce the flux, $\boldsymbol{\sigma} := \psi(|\nabla \mathbf{u}|)\,\nabla \mathbf{u} - p\,\mathbf{I}$, and the tensor gradient of the velocity, $\mathbf{t} := \nabla \mathbf{u}$, as additional unknowns. Let us denote by $\boldsymbol{\psi}(\mathbf{r}) := (\psi(|\mathbf{r}|)\, r_{ij})$, for all $\mathbf{r} \in \mathbb{R}^{2\times 2}$. Then, the nonlinear constitutive law and the equilibrium equation become, respectively,

$$\boldsymbol{\sigma} \,=\, \boldsymbol{\psi}(\mathbf{t}) \,-\, p\,\mathbf{I} \quad \text{and} \quad -\,\mathbf{div}(\boldsymbol{\sigma}) \,=\, \mathbf{f} \qquad \text{in } \Omega. \tag{15}$$

In addition, since $\mathrm{div}(\mathbf{u}) = \mathrm{tr}(\nabla \mathbf{u})$, the incompressibility condition can be rewritten as $\mathrm{tr}(\mathbf{t}) = 0$ in Ω. Multiplying the relation $\mathbf{t} = \nabla \mathbf{u}$ by a tensor

$\boldsymbol{\tau}$, integrating by parts, using that $\mathbf{u} = \mathbf{g}$ on Γ and testing appropriately the equations in (15) and the incompressibility condition, we obtain the following mixed variational formulation of (13): find $(\mathbf{t}, \boldsymbol{\sigma}, p, \mathbf{u}, \xi) \in [L^2(\Omega)]^{2\times 2} \times H(\mathbf{div};\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2 \times \mathbb{R}$ such that

$$\begin{aligned} \int_\Omega \boldsymbol{\psi}(\mathbf{t}) : \mathbf{s} - \int_\Omega \boldsymbol{\sigma} : \mathbf{s} - \int_\Omega p \,\mathrm{tr}(\mathbf{s}) &= 0, \\ -\int_\Omega \boldsymbol{\tau} : \mathbf{t} - \int_\Omega q \,\mathrm{tr}(\mathbf{t}) - \int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) + \xi \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) &= -\langle \boldsymbol{\tau}\mathbf{n}, \mathbf{g}\rangle_\Gamma, \\ -\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) + \eta \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) &= \int_\Omega \mathbf{f} \cdot \mathbf{v}, \end{aligned} \tag{16}$$

for all $(\mathbf{s}, \boldsymbol{\tau}, q, \mathbf{v}, \eta) \in [L^2(\Omega)]^{2\times 2} \times H(\mathbf{div};\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2 \times \mathbb{R}$. We introduce in (16) the additional unknown ξ, which is a Lagrange multiplier associated with the restriction $\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) = 0$, added to ensure uniqueness (see [8]). Actually, we know in advance that $\xi = 0$, but we keep this artificial unknown to ensure the symmetry of the whole formulation.

Next, we remark that (16) has a twofold saddle point structure. Indeed, let us introduce the spaces $X_1 := [L^2(\Omega)]^{2\times 2}$, $M_1 := H(\mathbf{div};\Omega) \times L^2(\Omega)$ and $M := [L^2(\Omega)]^2 \times \mathbb{R}$, and define the operators $A_1 : X_1 \to X_1'$, $B_1 : X_1 \to M_1'$ and $B : M_1 \to M'$ as follows:

$$[A_1(\mathbf{r}), \mathbf{s}] := \int_\Omega \boldsymbol{\psi}(\mathbf{r}) : \mathbf{s}\,, \quad [B_1(\mathbf{r}), (\boldsymbol{\tau}, q)] := -\int_\Omega \boldsymbol{\tau} : \mathbf{r} - \int_\Omega q \,\mathrm{tr}(\mathbf{r}),$$

$$[B(\boldsymbol{\tau}, q), (\mathbf{v}, \eta)] := -\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) + \eta \int_\Omega \mathrm{tr}(\boldsymbol{\tau}),$$

for all $\mathbf{r}, \mathbf{s} \in X_1$, $(\boldsymbol{\tau}, q) \in M_1$ and $(\mathbf{v}, \eta) \in M$, where $[\cdot,\cdot]$ stands for the duality pairing induced by the corresponding operators. Then, with the previous definitions for A_1, B_1 and B, (16) can be written as an operator equation with a matrix operator of the form (12). Under suitable assumptions on the nonlinear kinematic viscosity function ψ (see equations (1.2) and (1.3) in [29]), we proved that the continuous formulation (16) is well posed. The proof reduces to show that the hypotheses of Theorem 2.4 in [26] are satisfied.

In order to define the corresponding mixed finite element scheme, we assume for simplicity that Γ is a polygonal curve, and let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$ by triangles T of diameter h_T such that $h := \max\{\, h_T : T \in \mathcal{T}_h \,\}$ and $\overline{\Omega} = \cup\{\, T : T \in \mathcal{T}_h \,\}$. For each $T \in \mathcal{T}_h$, we let $\mathcal{RT}_0(T)$ be the local Raviart-Thomas space of lowest order and, for any non-negative integer k, we denote by $\mathcal{P}_k(T)$ the space of polynomials defined on T of degree $\le k$. Then, we introduce the following finite element subspaces:

$$X_{1,h} := \left\{ \mathbf{s} \in [L^2(\Omega)]^{2\times 2} : \mathbf{s}|_T \in [\mathcal{P}_0(T)]^{2\times 2} \quad \forall\, T \in \mathcal{T}_h \right\},$$

$$M_{1,h}^{\boldsymbol{\sigma}} := \{\boldsymbol{\tau} := (\tau_{ij}) \in H(\mathbf{div};\Omega) : (\tau_{i1}\ \tau_{i2})^{\mathrm{t}}|_T \in \mathcal{RT}_0(T)\ i = 1,2\,,\ \forall\, T \in \mathcal{T}_h\},$$

$$M_{1,h}^{p} := \{\, q \in L^2(\Omega) : q|_T \in \mathcal{P}_0(T) \quad \forall\, T \in \mathcal{T}_h \,\},$$

$$M_h^{\mathbf{u}} := \{\, \mathbf{v} \in [L^2(\Omega)]^2 \,:\, \mathbf{v}|_T \in [\mathcal{P}_0(T)]^2 \quad \forall\, T \in \mathcal{T}_h \,\}.$$

We showed that the corresponding Galerkin scheme has a unique solution $(\mathbf{t}_h, \boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \xi_h) \in X_{1,h} \times M_{1,h}^{\boldsymbol{\sigma}} \times M_{1,h}^{p} \times M_h^{\mathbf{u}} \times \mathbb{R}$. Furthermore, using a Céa estimate and the approximation properties of the subspaces $X_{1,h}$, $M_{1,h}^{\boldsymbol{\sigma}}$, $M_{1,h}^{p}$ and $M_h^{\mathbf{u}}$, that follow from classical error estimates for projection and equilibrium interpolation operators (see e.g. [63]), we obtained the following rate of convergence. If $\mathbf{t} \in [H^1(\Omega)]^{2\times 2}$, $\boldsymbol{\sigma} \in [H^1(\Omega)]^{2\times 2}$, $\mathbf{div}(\boldsymbol{\sigma}) \in [H^1(\Omega)]^2$, $p \in H^1(\Omega)$ and $\mathbf{u} \in [H^1(\Omega)]^2$, then there exists $C > 0$, independent of h, such that

$$\|(\mathbf{t}, \boldsymbol{\sigma}, p, \mathbf{u}, \xi) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \xi_h)\| \,\leq\, C\, h \Big(\|\mathbf{t}\|_{[H^1(\Omega)]^{2\times 2}} + \\ + \|\boldsymbol{\sigma}\|_{[H^1(\Omega)]^{2\times 2}} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{[H^1(\Omega)]^2} + \|p\|_{H^1(\Omega)} + \|\mathbf{u}\|_{[H^1(\Omega)]^2} \Big).$$

Recently, V.J. Ervin et al. [24] recasted the formulation introduced in [29] in appropriate Sobolev spaces, providing tighter error estimates for the approximate solution and showing that higher-order approximating spaces can be used.

On the other hand, the application of adaptive algorithms based on a posteriori error estimates usually guarantees the quasi-optimal rate of convergence of the finite element solution to a boundary value problem. These techniques are specially useful for nonlinear problems, where no a priori hints on how to build suitable meshes are available. As shown in [3, 40], the combination of the usual Bank-Weiser approach from [2] with the analysis from [9] and [10] allows to derive fully explicit and reliable a posteriori error estimates for dual-dual variational formulations. In [30] we followed [3] and obtained reliable and quasi-efficient a posteriori error estimators for the nonlinear Stokes problems analyzed in [29]. Numerical experiments illustrate the performance of the mixed finite element scheme and confirm the reliability and quasi-efficiency of the a posteriori error estimators. They also show that the associated adaptive algorithm is much more efficient than a uniform refinement procedure.

As we mentioned before, in [11] we applied the approach from [29] to derive a low-order mixed FEM for the generalized Stokes problem. The generalized Stokes problem is a Stokes-like linear system with a dominating zeroth order term. This problem arises naturally in the time discretization of the corresponding non-steady equations and hence, plays a fundamental role in the numerical simulation of viscous incompressible flows (laminar and turbulent). Indeed, the most expensive part of the solution procedure for the time-dependent Navier-Stokes equations reduces to solve the generalized Stokes problem at each nonlinear iteration. Given $\mathbf{f} \in [L^2(\Omega)]^2$ and $\mathbf{g} \in [H^{1/2}(\Gamma)]^2$, we look for the velocity $\mathbf{u} := (u_1, u_2)^{\mathtt{t}}$ and the pressure p of a fluid occupying the region Ω, and such that

$$\left\{ \begin{array}{rcll} \alpha\, \mathbf{u} - \nu\, \Delta \mathbf{u} + \nabla p &=& \mathbf{f} & \text{in } \Omega, \\ \mathrm{div}(\mathbf{u}) &=& 0 & \text{in } \Omega, \\ \mathbf{u} &=& \mathbf{g} & \text{on } \Gamma. \end{array} \right. \tag{17}$$

where $\nu > 0$ is the kinematic viscosity of the fluid, that we assume constant, and α is a positive parameter proportional to the inverse of the time-step (we may assume, without loss of generality, that $\alpha \geq \nu$). We recall that, due to the incompressibility of the fluid, the Dirichlet datum $\mathbf{g}$ must satisfy the compatibility condition (14).

Now, we proceed as in [29] and introduce the tensor gradient of the velocity $\mathbf{t} := \nabla \mathbf{u}$ and the flux $\boldsymbol{\sigma} := \nu \nabla \mathbf{u} - p\,\mathbf{I}$ as additional unknowns in Ω. In this way, we obtain a mixed variational formulation of problem (17) that shows a twofold saddle point structure (see (12)). Indeed, let us define the spaces $X_1 := [L^2(\Omega)]^{2\times 2} \times [L^2(\Omega)]^2$, $M_1 := H(\mathbf{div};\Omega)$, $X := X_1 \times M_1$ and $M := L^2(\Omega) \times \mathbb{R}$, and the operators $A_1 : X_1 \to X_1'$, $B_1 : X_1 \to M_1'$ and $B : X \to M$ as follows:

$$[A_1(\mathbf{s},\mathbf{v}),(\mathbf{r},\mathbf{w})] := \nu \int_\Omega \mathbf{s} : \mathbf{r} + \alpha \int_\Omega \mathbf{v}\cdot\mathbf{w},$$
$$[B_1(\mathbf{s},\mathbf{v}),\boldsymbol{\tau}] := -\int_\Omega \boldsymbol{\tau} : \mathbf{s} - \int_\Omega \mathbf{div}(\boldsymbol{\tau})\cdot\mathbf{v},$$
$$[B(\mathbf{s},\mathbf{v},\boldsymbol{\tau}),(q,\eta)] := -\int_\Omega q\,\mathrm{tr}(\mathbf{s}) + \eta \int_\Omega \mathrm{tr}(\boldsymbol{\tau}),$$

for all $(\mathbf{s},\mathbf{v}),(\mathbf{r},\mathbf{w}) \in X_1$, $\boldsymbol{\tau} \in M_1$ and $(q,\eta) \in M$. Then, the variational formulation of (17) can be set equivalently as: find $((\mathbf{t},\mathbf{u},\boldsymbol{\sigma}),(p,\xi)) \in X \times M$ such that

$$\begin{pmatrix} A & B^* \\ B & O \end{pmatrix} \begin{pmatrix} (\mathbf{t},\mathbf{u},\boldsymbol{\sigma}) \\ (p,\xi) \end{pmatrix} = \begin{pmatrix} F \\ O \end{pmatrix}, \tag{18}$$

where A is defined as in (12) and $[F,(\mathbf{s},\mathbf{v},\boldsymbol{\tau})] := \int_\Omega \mathbf{f}\cdot\mathbf{v} - \langle \boldsymbol{\tau}\mathbf{n},\mathbf{g}\rangle_\Gamma$. Using the abstract theory from [26], we proved that problem (18) has a unique solution $((\mathbf{t},\mathbf{u},\boldsymbol{\sigma}),(p,\xi)) \in X \times M$, and that there exists a positive constant $C(\alpha,\nu) = \mathcal{O}(\frac{\alpha^3}{\nu})$ such that

$$\|((\mathbf{t},\mathbf{u},\boldsymbol{\sigma}),(p,\xi))\|_{X\times M} \;\leq\; C(\alpha,\nu)\;\left(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{g}\|_{H^{1/2}(\Gamma)}\right). \tag{19}$$

The corresponding Galerkin scheme is defined using the same finite element subspaces as in [29] for the unknowns $\mathbf{u}$, $\boldsymbol{\sigma}$ and p. However, to guarantee stability, the approximating space of the tensor gradient of the velocity, $\mathbf{t}$, has to be suitably enriched. Indeed, we have to include the deviator of the vector Raviart-Thomas space of lowest order, that is, we define

$$X_{1,h}^{\mathbf{t}} := \{\mathbf{s} \in [L^2(\Omega)]^{2\times 2} \,:\, \mathbf{s}|_T \in \mathcal{A}_0(T) \quad \forall\, T \in \mathcal{T}_h\},$$

where

$$\mathcal{A}_0(T) := [\mathcal{P}_0(T)]^{2\times 2} \oplus \left\langle \left\{ \begin{pmatrix} x_1 & 2x_2 \\ 0 & -x_1 \end{pmatrix}, \begin{pmatrix} -x_2 & 0 \\ 2x_1 & x_2 \end{pmatrix} \right\} \right\rangle.$$

We proved that the discrete scheme is well-posed and derived the corresponding rate of convergence, in which a constant $\bar{C}(\alpha,\nu) = \mathcal{O}(\frac{\alpha^3}{\nu})$ is involved.

Theoretical results suggest that the rate of convergence is affected by large values of α, which, nevertheless, is not too severe in the numerical experiments (see [11]). Further, since α is proportional to the inverse of the time-step, Δt, the estimates also lead us to think that the convergence of time-dependent solutions should deteriorate as Δt decreases.

We followed [3, 30] and developed an a posteriori error analysis based on local problems. In this way, we obtained reliable and quasi-efficient a posteriori error estimators, that depend on the choice of two auxiliary functions. Numerical experiments confirm the reliability and quasi-efficiency of the estimators and illustrate the ability of the associated adaptive algorithm to localize the boundary layers, inner layers and singularities of the solution. Moreover, according to the theory, one would expect effectivity indexes between $\mathcal{O}(\alpha^{-1})$ and $\mathcal{O}(\frac{\alpha^3}{\nu})$. However, we observe in practice that they all lie on ranges much tighter than that, they do not deteriorate as the number of degrees of freedom increases and, in addition, they improve from uniform to adaptive refinements. The above observations yield the conjecture that these constants are overestimated. To conclude, this mixed method is perhaps not so competitive for extremely large values of α, but constitutes a good alternative for moderately large values of this parameter. Numerical experiments show that the adaptive algorithm is much more efficient than a uniform refinement when solving the discrete scheme.

Most approaches to the problems considered in [29, 11] deal with the usual pressure-velocity formulation, in which the velocity lives in $[H^1(\Omega)]^2$. This means, in particular, that the finite element subspace for the velocity needs to be a subset of the continuous functions. In addition, the Dirichlet boundary condition, being essential and non-homogeneous, cannot be incorporated either in the continuous and discrete formulations or in the definitions of the spaces involved, and therefore one is necessarily led to a non-conforming Galerkin scheme (certainly, we refer to the theoretical analysis of the method, since the interpolation of essential boundary conditions does not cause any difficulty in practice). In turn, in a dual-mixed setting the velocity becomes an unknown in $[L^2(\Omega)]^2$, which gives more flexibility to choose the associated finite element subspace (for instance, piecewise constant functions are a feasible choice). Furthermore, the Dirichlet boundary condition, being now natural, is incorporated directly into the right hand sides of the continuous and discrete formulations and hence, we avoid the error analysis of a non-conforming scheme.

Another important advantage of using dual-mixed methods, already pointed out, is the possibility of introducing further unknowns of physical interest (like the flux). These unknowns are then approximated directly, avoiding any numerical postprocessing that could yield additional sources of error. Moreover, the conservativity properties are transferred to some of these unknowns (for instance, continuity of the normal components of the flux), which, as we have seen, can be approximated with finite elements of very low order as well. Finally, we recall that the derivation of finite element subspaces guaranteeing unique solvability and stability of the Galerkin schemes for dual-dual formulations in

elasticity and fluid mechanics was unified in [12].

4 A posteriori error analysis of augmented mixed finite element methods in elasticity

In this section we consider the augmented mixed finite element method introduced in [27] for the linear elasticity system in the plane and outline a residual-based a posteriori error analysis developed in [4] in the case of pure homogeneous Dirichlet boundary conditions. The analysis in the case of mixed boundary conditions can be found in [5].

Let Ω be a simply connected domain in $\mathbb{R}^2$. We assume, for simplicity, that Ω has a polygonal boundary Γ. Given a volume force $\mathbf{f} \in [L^2(\Omega)]^2$, we consider the problem of computing the displacements $\mathbf{u}$ and the stress tensor $\boldsymbol{\sigma}$ of a linear elastic material occupying the region Ω and such that

$$\left\{ \begin{array}{rcll} \boldsymbol{\sigma} & = & \mathcal{C}\,\mathbf{e}(\mathbf{u}) & \text{in } \Omega, \\ -\,\mathbf{div}(\boldsymbol{\sigma}) & = & \mathbf{f} & \text{in } \Omega, \\ \mathbf{u} & = & \mathbf{0} & \text{on } \Gamma. \end{array} \right. \tag{20}$$

Hereafter, $\mathbf{e}(\mathbf{u}) := \frac{1}{2}\,(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathtt{t}})$ is the strain tensor of small deformations and $\mathcal{C}$ is the elasticity tensor determined by Hooke's law:

$$\mathcal{C}\,\boldsymbol{\zeta} \;:=\; \lambda\,\mathrm{tr}(\boldsymbol{\zeta})\,\mathbf{I} \;+\; 2\,\mu\,\boldsymbol{\zeta} \qquad \forall\,\boldsymbol{\zeta} \,\in\, [L^2(\Omega)]^{2\times 2}\,, \tag{21}$$

where λ, $\mu > 0$ are the Lamé constants.

Recently, a new stabilized mixed finite element method for plane linear elasticity was presented and analyzed in [27]. The approach is based on the introduction of suitable Galerkin least-squares terms arising from the constitutive and equilibrium equations, and from the relation defining the rotation $\boldsymbol{\gamma}$ in terms of the displacement, $\boldsymbol{\gamma} := \frac{1}{2}\,(\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathtt{t}})$. In particular, given positive parameters, κ_1, κ_2 and κ_3, independent of λ, the following augmented variational formulation for problem (20) is proposed: find $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) \,\in\, \hat{H}_0 \,:=\, H_0 \times [H_0^1(\Omega)]^2 \times [L^2(\Omega)]^{2\times 2}_{\mathtt{skew}}$ such that

$$A((\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})) \;=\; F(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \qquad \forall\,(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \in \hat{H}_0, \tag{22}$$

where $[L^2(\Omega)]^{2\times 2}_{\mathtt{skew}} := \Big\{\, \boldsymbol{\eta} \in [L^2(\Omega)]^{2\times 2} \,:\, \boldsymbol{\eta} + \boldsymbol{\eta}^{\mathtt{t}} \,=\, \mathbf{0} \,\Big\}$,

$$H_0 := \left\{ \boldsymbol{\tau} \in H(\mathbf{div}; \Omega) \,:\, \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \,=\, 0 \right\},$$

and the bilinear form $A : \hat{H}_0 \times \hat{H}_0 \,\to\, \mathbb{R}$ and the functional $F : \hat{H}_0 \,\to\, \mathbb{R}$ are

defined by

$$
\begin{aligned}
&A((\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma}),(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta})) := \\
&:= \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} - \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) - \int_\Omega \boldsymbol{\eta} : \boldsymbol{\sigma} \\
&+ \kappa_1 \int_\Omega \big(\,\mathbf{e}(\mathbf{u}) - \mathcal{C}^{-1}\,\boldsymbol{\sigma}\,\big) : \big(\,\mathbf{e}(\mathbf{v}) + \mathcal{C}^{-1}\,\boldsymbol{\tau}\,\big) \;+\; \kappa_2 \int_\Omega \mathbf{div}(\boldsymbol{\sigma})\cdot\mathbf{div}(\boldsymbol{\tau}) \\
&+ \kappa_3 \int_\Omega \left(\boldsymbol{\gamma} - \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}})\right) : \left(\boldsymbol{\eta} + \frac{1}{2}(\nabla\mathbf{v} - (\nabla\mathbf{v})^{\mathtt{t}})\right),
\end{aligned}
$$

and

$$
F(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}) \;:=\; \int_\Omega \mathbf{f}\cdot(\,\mathbf{v} \;-\; \kappa_2\,\mathbf{div}(\boldsymbol{\tau})\,)\;.
$$

We recall that it is easy to see from (21) that the inverse tensor $\mathcal{C}^{-1}$ reduces to

$$
\mathcal{C}^{-1}\,\boldsymbol{\zeta} \;:=\; \frac{1}{2\,\mu}\,\boldsymbol{\zeta} \;-\; \frac{\lambda}{4\,\mu\,(\lambda+\mu)}\,\mathrm{tr}(\boldsymbol{\zeta})\,\mathbf{I} \qquad \forall\,\boldsymbol{\zeta} \;\in\; [L^2(\Omega)]^{2\times 2}.
$$

Assume that $(\kappa_1,\kappa_2,\kappa_3)$ is independent of λ and such that

$$
0 < \kappa_3 < \kappa_1 < 2\,\mu \qquad \text{and} \qquad 0 < \kappa_2. \tag{23}
$$

Then, the bilinear form $A(\cdot,\cdot)$ is strongly coercive and continuous, and therefore, problem (22) is well-posed (see Theorems 3.1 and 3.2 in [27]). In particular, if we take $\kappa_2 = \frac{1}{\mu}\left(1 - \frac{\kappa_1}{2\,\mu}\right)$, then the stability constant depends only on μ, $\dfrac{1}{\mu}$ and Ω.

The augmented variational formulation (22), being strongly coercive, allows to use arbitrary finite element subspaces to define the corresponding discrete scheme. This constitutes one of its main advantages, as compared with the traditional mixed finite element schemes for the linear elasticity problem (see [8]). Indeed, given a finite element subspace $\hat{H}_{0,h} := H^{\boldsymbol{\sigma}}_{0,h} \times H^{\mathbf{u}}_{0,h} \times H^{\boldsymbol{\gamma}}_{0,h} \subseteq \hat{H}_0$, the Galerkin scheme associated to (22) reads: find $(\boldsymbol{\sigma}_h,\mathbf{u}_h,\boldsymbol{\gamma}_h) \in \hat{H}_{0,h}$ such that

$$
A((\boldsymbol{\sigma}_h,\mathbf{u}_h,\boldsymbol{\gamma}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h,\boldsymbol{\eta}_h)) = F(\boldsymbol{\tau}_h,\mathbf{v}_h,\boldsymbol{\eta}_h) \qquad \forall\,(\boldsymbol{\tau}_h,\mathbf{v}_h,\boldsymbol{\eta}_h) \in \hat{H}_{0,h}. \tag{24}
$$

If the parameters κ_1, κ_2 and κ_3 satisfy (23), then the discrete problem (24) is well-posed for any arbitrary choice of the subspace $\hat{H}_{0,h}$. In particular, it is possible to use Raviart-Thomas spaces of lowest order to approximate the stress tensor $\boldsymbol{\sigma}$, piecewise linear elements for the displacement $\mathbf{u}$, and piecewise constants for the rotation $\boldsymbol{\gamma}$. The rate of convergence of (24) for this specific finite element subspace is given in Theorem 4.2 in [27].

As compared with more traditional mixed methods, such as PEERS and BDM, and besides the fact of being able to choose any finite element subspace, the augmented approach presents other important advantages. Indeed, it

becomes a much cheaper alternative since the global number of degrees of freedom in terms of the number of triangles is much smaller (see [4, Section 5]). In addition, if we choose the finite element subspace of the lowest order, the augmented scheme (24) yields simpler computations.

The competitive character of the augmented mixed finite element method (24) motivated the derivation of a posteriori error estimators for this scheme. We need to introduce some notations. Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$ by triangles T of diameter h_T such that $h := \max\{\, h_T : T \in \mathcal{T}_h \,\}$ and $\overline{\Omega} = \cup\{\, T : T \in \mathcal{T}_h \,\}$. Given $T \in \mathcal{T}_h$, we let $E(T)$ be the set of its edges and let $E_h(\Omega)$ be the set of all interior edges of the triangulation $\mathcal{T}_h$. In what follows, h_e stands for the length of edge e. Further, given $\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}$ (such that $\boldsymbol{\tau}|_T \in \mathcal{C}(T)$ on each $T \in \mathcal{T}_h$), an edge $e \in E(T) \cap E_h(\Omega)$ and the unit tangential vector $\mathbf{t}$ along e, we denote by $J[\boldsymbol{\tau}\mathbf{t}]$ the tangential jump across e, that is, $J[\boldsymbol{\tau}\mathbf{t}] := (\boldsymbol{\tau}|_T - \boldsymbol{\tau}|_{T'})|_e\mathbf{t}$, where $T' \in \mathcal{T}_h$ is such that $T \cap T' = e$. We recall that $\mathbf{t} := (-n_2, n_1)^{\mathtt{t}}$, where $\mathbf{n} := (n_1, n_2)^{\mathtt{t}}$ is the unit outward normal to ∂T. The normal jumps $J[\boldsymbol{\tau}\mathbf{n}]$ are defined analogously.

Then, if $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) \in \hat{H}_0$ and $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \hat{H}_{0,h}$ are, respectively, the solutions to the continuous and discrete formulations, (22) and (24), we define the error indicator θ_T, for any $T \in \mathcal{T}_h$, as follows:

$$
\begin{aligned}
\theta_T^2 &:= \|\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|^2_{[L^2(T)]^2} + \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\|^2_{[L^2(T)]^{2\times 2}} \\
&+ \|\boldsymbol{\gamma}_h - \tfrac{1}{2}(\nabla \mathbf{u}_h - (\nabla \mathbf{u}_h)^{\mathtt{t}})\|^2_{[L^2(T)]^{2\times 2}} + h_T^2 \, \|\, \mathbf{curl}(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + \boldsymbol{\gamma}_h)\|^2_{[L^2(T)]^2} \\
&+ \sum_{e \in E(T)} h_e \, \|J[(\mathcal{C}^{-1}\boldsymbol{\sigma}_h - \nabla \mathbf{u}_h + \boldsymbol{\gamma}_h)\mathbf{t}]\|^2_{[L^2(e)]^2} \\
&+ h_T^2 \, \|\, \mathbf{curl}(\mathcal{C}^{-1}(\mathbf{e}(\mathbf{u}_h) - \mathcal{C}^{-1}\boldsymbol{\sigma}_h))\|^2_{[L^2(T)]^2} \\
&+ \sum_{e \in E(T)} h_e \, \|J[(\mathcal{C}^{-1}(\mathbf{e}(\mathbf{u}_h) - \mathcal{C}^{-1}\boldsymbol{\sigma}_h))\mathbf{t}]\|^2_{[L^2(e)]^2} \\
&+ h_T^2 \, \|\mathbf{div}(\mathbf{e}(\mathbf{u}_h) - \tfrac{1}{2}(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + (\mathcal{C}^{-1}\boldsymbol{\sigma}_h)^{\mathtt{t}}))\|^2_{[L^2(T)]^2} \\
&+ \sum_{e \in E(T) \cap E_h(\Omega)} h_e \, \|J[(\mathbf{e}(\mathbf{u}_h) - \frac{1}{2}(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + (\mathcal{C}^{-1}\boldsymbol{\sigma}_h)^{\mathtt{t}}))\mathbf{n}]\|^2_{[L^2(e)]^2} \\
&+ h_T^2 \, \|\mathbf{div}(\boldsymbol{\gamma}_h - \tfrac{1}{2}(\nabla \mathbf{u}_h - (\nabla \mathbf{u}_h)^{\mathtt{t}}))\|^2_{[L^2(T)]^2} \\
&+ \sum_{e \in E(T) \cap E_h(\Omega)} h_e \, \|J[(\boldsymbol{\gamma}_h - \frac{1}{2}(\nabla \mathbf{u}_h - (\nabla \mathbf{u}_h)^{\mathtt{t}}))\mathbf{n}]\|^2_{[L^2(e)]^2} \, .
\end{aligned}
$$

The residual character of each term involved in the definition of θ_T is quite clear. In addition, we observe that some of these terms are known from residual estimators for the usual (non-augmented) mixed finite element method in linear elasticity (see, e.g. [15]). However, most of them are new since they arise from the new Galerkin least-squares terms introduced in (22). Finally, we remark that when $\boldsymbol{\sigma}_h|_T \in [\mathcal{RT}_0(T)^{\mathtt{t}}]^2$, $\mathbf{u}_h|_T \in [\mathcal{P}_1(T)]^2$ and $\boldsymbol{\gamma}_h|_T \in [\mathcal{P}_0(T)]^{2\times 2}$, some of the terms in the definition of θ_T vanish.

As usual, $\theta := \left(\sum_{T \in \mathcal{T}_h} \theta_T^2 \right)^{1/2}$ is used as the global residual error estimator. We proved in [4] that the a posteriori error estimator θ is reliable and efficient, that is, there exist $C_{\mathtt{eff}}, C_{\mathtt{rel}} > 0$, independent of h and λ, such that

$$C_{\mathtt{eff}}\, \theta \;\leq\; \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_{\hat{H}_0} \;\leq\; C_{\mathtt{rel}}\, \theta\,. \tag{25}$$

Reliability (upper bound in (25)) ensures that we obtain a numerical solution with an accuracy below a prescribed tolerance. Local lower bounds are necessary to ensure that the mesh is correctly refined so that one obtains a numerical solution with a prescribed tolerance using a (nearly) minimal number of nodes.

To prove that θ is reliable, we combined a technique used in mixed finite element schemes (see, e.g. [14, 15]) with the usual procedure applied to primal finite element methods (see [69]). It is important to remark that just one of these approaches by itself would not be enough in this case. Up to our knowledge, this combined analysis seems to be applied in [4] for the first time. We provide next a sketch of the proof.

We consider the following auxiliary problem: find $\mathbf{z} \in [H_0^1(\Omega)]^2$ such that

$$\left\{ \begin{array}{rcll} -\mathbf{div}(\mathbf{e}(\mathbf{z})) & = & \mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h) & \text{in } \Omega, \\ \mathbf{z} & = & \mathbf{0} & \text{on } \Gamma, \end{array} \right. \tag{26}$$

and define $\boldsymbol{\sigma}^* := \mathbf{e}(\mathbf{z})$, where $\mathbf{z}$ is the unique solution to problem (26). It follows that $\boldsymbol{\sigma}^* \in H_0$ and, because of the continuous dependence result, there exists $c > 0$ such that

$$\|\boldsymbol{\sigma}^*\|_{H(\mathbf{div};\Omega)} \;\leq\; c\, \|\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|_{[L^2(\Omega)]^2}. \tag{27}$$

In addition, it is easy to see that $\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h - \boldsymbol{\sigma}^*) = \mathbf{0}$ in Ω. Then, using the triangle inequality, that A is coercive and bounded, and (27), we obtain that there exists $C > 0$, independent of h and λ, such that

$$\begin{aligned} C\, \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_{\hat{H}_0} \leq\; & \|\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|_{[L^2(\Omega)]^2} + \\ & + \sup_{\substack{\mathbf{0} \neq (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \in \hat{H}_0 \\ \mathbf{div}(\boldsymbol{\tau}) = \mathbf{0}}} \frac{A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}))}{\|(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})\|_{\hat{H}_0}}\,. \end{aligned} \tag{28}$$

It remains to bound the second term on the right hand side of (28). To this end, we make use of the well-known Clément interpolation operator, $I_h : H^1(\Omega) \to X_h$, where X_h is the space of continuous piecewise linear functions on $\mathcal{T}_h$, which satisfies the standard local approximation properties stated in [20]. We recall that $I_h(v) \in X_h \cap H_0^1(\Omega)$ for all $v \in H_0^1(\Omega)$.

Now, let $(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \in \hat{H}_0$, $(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \neq \mathbf{0}$, be such that $\mathbf{div}(\boldsymbol{\tau}) = \mathbf{0}$ in Ω. Since we assume the domain Ω to be simply connected, there exists a stream function $\boldsymbol{\varphi} := (\varphi_1, \varphi_2) \in [H^1(\Omega)]^2$ such that $\int_\Omega \varphi_i = 0$, for $i = 1, 2$, and $\boldsymbol{\tau} = \underline{\mathbf{curl}}(\boldsymbol{\varphi})$. Then, we define $\boldsymbol{\varphi}_h := (\varphi_{1,h}, \varphi_{2,h})$, with $\varphi_{i,h} := I_h(\varphi_i)$ for

$i = 1, 2$, and $\boldsymbol{\tau}_h := \underline{\mathbf{curl}}(\boldsymbol{\varphi}_h)$. Note that we can write $\boldsymbol{\tau}_h = \boldsymbol{\tau}_{h,0} + d_h\,\mathbf{I}$, where $\boldsymbol{\tau}_{h,0} \in H^{\boldsymbol{\sigma}}_{0,h}$ and $d_h = \frac{\int_\Omega \mathrm{tr}(\boldsymbol{\tau}_h)}{2|\Omega|} \in \mathbb{R}$.

On the other hand, an immediate consequence of (22) and (24) is the Galerkin orthogonality:

$$A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h)) = 0 \quad \forall\, (\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \in \hat{H}_{0,h}\,. \tag{29}$$

Let $\mathbf{v}_h := (I_h(v_1), I_h(v_2)) \in H^{\mathbf{u}}_{0,h}$ be the vector Clément interpolant of $\mathbf{v} := (v_1, v_2) \in [H^1_0(\Omega)]^2$. Then, it follows from (29) that

$$\begin{aligned} &A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})) = \\ &\qquad = A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h), (\boldsymbol{\tau} - \boldsymbol{\tau}_{h,0}, \mathbf{v} - \mathbf{v}_h, \boldsymbol{\eta})). \end{aligned} \tag{30}$$

Since $\int_\Omega \mathrm{tr}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) = 0$ and $\mathbf{u} - \mathbf{u}_h = \mathbf{0}$ on Γ, using the orthogonality between symmetric and skew-symmetric tensors, we obtain that

$$A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h), (d_h\mathbf{I}, \mathbf{0}, \mathbf{0})) = 0.$$

Hence, from (30) and (22) we deduce that

$$\begin{aligned} &A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})) \;= \\ &\quad = \; F(\boldsymbol{\tau} - \boldsymbol{\tau}_h, \mathbf{v} - \mathbf{v}_h, \boldsymbol{\eta}) \;-\; A((\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h), (\boldsymbol{\tau} - \boldsymbol{\tau}_h, \mathbf{v} - \mathbf{v}_h, \boldsymbol{\eta}))\,. \end{aligned}$$

According to the definitions of the forms $A(\cdot,\cdot)$ and $F(\cdot)$, taking into account that $\mathbf{div}(\boldsymbol{\tau} - \boldsymbol{\tau}_h) = \mathbf{div}(\underline{\mathbf{curl}}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)) = \mathbf{0}$ and using again (29), we find (after some algebraic manipulations) that

$$\begin{aligned} A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})) \;&=\; \int_\Omega (\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)) \cdot (\mathbf{v} - \mathbf{v}_h) \\ &+ \frac{1}{2} \int_\Omega (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}) : \boldsymbol{\eta} \;-\; \kappa_3 \int_\Omega \left(\boldsymbol{\gamma}_h - \frac{1}{2}(\nabla \mathbf{u}_h - (\nabla \mathbf{u}_h)^{\mathtt{t}}) \right) : \boldsymbol{\eta} \\ &- \int_\Omega \left((\mathcal{C}^{-1}\boldsymbol{\sigma}_h - \nabla \mathbf{u}_h + \boldsymbol{\gamma}_h) \;+\; \kappa_1\, \mathcal{C}^{-1}(\mathbf{e}(\mathbf{u}_h) - \mathcal{C}^{-1}\boldsymbol{\sigma}_h) \right) : (\boldsymbol{\tau} - \boldsymbol{\tau}_h) \\ &- \kappa_1 \int_\Omega \left(\mathbf{e}(\mathbf{u}_h) - \frac{1}{2}(\mathcal{C}^{-1}\boldsymbol{\sigma}_h + (\mathcal{C}^{-1}\boldsymbol{\sigma}_h)^{\mathtt{t}}) \right) : \nabla(\mathbf{v} - \mathbf{v}_h) \\ &+ \kappa_3 \int_\Omega \left(\boldsymbol{\gamma}_h - \frac{1}{2}(\nabla \mathbf{u}_h - (\nabla \mathbf{u}_h)^{\mathtt{t}}) \right) : \nabla(\mathbf{v} - \mathbf{v}_h). \end{aligned} \tag{31}$$

The rest of the proof of reliability consists in deriving suitable upper bounds for each one of the terms appearing on the right hand side of (31); we omit the details and refer the reader to Section 3 in [4].

On the other hand, to show that the a posteriori error estimator θ is efficient (lower bound in (25)), we proceed as in [14] and [15] and apply inverse inequalities and the localization technique introduced in [69], which is based on

triangle-bubble and edge-bubble functions (see Section 4 in [4] for more details). We remark that, because of the new terms in the definition of θ (those involving the **curl** and **div** operators and the normal and tangential jumps across the edges of the triangulation), we needed to establish more general versions of some technical lemmas concerning inverse estimates and piecewise polynomials (see Lemmas 4.3-4.6 in [4]). The generality of these results allows to eventually apply them not only in the present context, but also in the a posteriori error analysis of other primal and mixed finite element methods.

We proposed the following adaptive algorithm, based on the a posteriori error estimator θ, to compute the solutions of (24) (cf. [69]):

1. Start with a coarse mesh $\mathcal{T}_h$.
2. Solve the Galerkin scheme (24) for the current mesh $\mathcal{T}_h$.
3. Compute θ_T for each triangle $T \in \mathcal{T}_h$.
4. Consider stopping criterion and decide to finish or go to next step.
5. Use *blue-green* procedure to refine each element $T' \in \mathcal{T}_h$ such that
$$\theta'_T \geq \frac{1}{2} \max_{T \in \mathcal{T}_h} \theta_T.$$
6. Define resulting mesh as the new $\mathcal{T}_h$ and go to step 2.

Numerical experiments underline the reliability and efficiency of the a posteriori error estimator θ and strongly demonstrate that the associated adaptive algorithm is much more suitable than a uniform discretization procedure when solving problems with non-smooth solutions. The robustness of θ with respect to the Poisson ratio and the ability of the adaptive algorithm to localize the singularities and large stress regions of the solution are also illustrated.

We recognize that θ is certainly more expensive than, for instance, the error indicator introduced in [6]. However, it is clear that the reliability and efficiency of θ become more advantageous features than the sole reliability of the estimator from [6]. Finally, in connection with the residual-based a posteriori error estimator developed in [15] for PEERS and BDM, which is also reliable and efficient, we point out that the advantage of θ, though a bit more expensive, is still the freedom to choose the finite element subspaces in the augmented scheme (24).

Finally, we mention that we have introduced an augmented primal-mixed method for the linear elasticity problem in the plane (see [28]) that involves four unknowns, namely: the displacement, the stress tensor, the strain tensor of small deformations and the pressure. This new variational formulation relies on the Hu-Washizu principle and was obtained by adding a least-squares term that involves the strain tensor of small deformations. We established sufficient conditions for the well-posedness of the corresponding Galerkin scheme and described a way to obtain stable finite element subspaces from any stable pair for the Stokes problem. Error estimates are also provided.

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